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# Probabilistic Construction of Solutions To Some Integral Equations

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## §1. Notations and assumptions

For simplicity, let  $D_0 := \mathbb{R}^3 \setminus \{0\}$ , and we put  $\mathbb{R}_+ := [0, \infty)$ . For every  $\alpha, \beta \in \mathbb{C}^3$ , we use the symbol  $\alpha \cdot \beta$  for the inner product, and we define  $e_x := x/|x|$  for every  $x \in D_0$ . In this article we consider the following deterministic nonlinear integral equation:

$$\begin{aligned} e^{\lambda t|x|^2} u(t, x) &= u_0(x) + \frac{\lambda}{2} \int_0^t ds e^{\lambda s|x|^2} \int p(s, x, y; u) n(x, y) dy \\ &\quad + \frac{\lambda}{2} \int_0^t e^{\lambda s|x|^2} f(s, x) ds, \quad \text{for } \forall (t, x) \in \mathbb{R}_+ \times D_0. \end{aligned} \quad (1)$$

Here  $u \equiv u(t, x)$  is an unknown function :  $\mathbb{R}_+ \times D_0 \rightarrow \mathbb{C}^3$ ,  $\lambda > 0$ , and  $u_0 : D_0 \rightarrow \mathbb{C}^3$  is the initial data such that  $u(t, x)|_{t=0} = u_0(x)$ . Moreover,  $f(t, x) : \mathbb{R}_+ \times D_0 \rightarrow \mathbb{C}^3$  is a given function satisfying  $f(t, x)/|x|^2 =: \tilde{f} \in L^1(\mathbb{R}_+)$ . The integrand  $p$  in (1) is given by

$$p(t, x, y; u) = u(x, y) \cdot e_x \{u(t, x - y) - e_x(u(t, x - y) \cdot e_x)\}. \quad (2)$$

On the other hand, we consider a Markov kernel  $K : D_0 \rightarrow D_0 \times D_0$ . Actually, for every  $z \in D_0$ ,  $K_z(dx, dy)$  lies in the space  $\mathcal{P}(D_0 \times D_0)$  of all probability measures on a product space  $D_0 \times D_0$ . When the kernel  $k$  is given by  $k(x, y) = i|x|^{-2}n(x, y)$ , then we define  $K_z$  as a Markov kernel satisfying that for any positive measurable function  $h = h(x, y)$  on  $D_0 \times D_0$ ,

$$\iint h(x, y) K_z(dx, dy) = \int h(x, z - x) k(x, z) dx. \quad (3)$$

Moreover, we assume that for every measurable functions  $f, g > 0$  on  $\mathbb{R}^+$ ,

$$\int h(|z|) \nu(dz) \int g(|x|) K_z(dx, dy) = \int g(|z|) \nu(dz) \int h(|y|) K_z(dx, dy) \quad (4)$$

holds, where the measure  $\nu$  is given by  $\nu(dz) = |z|^{-3} dz$ .

## §2. Main result

In this section we shall state our main result, which asserts the existence and uniqueness of solutions to the nonlinear integral equation (1). As a matter of fact, the solution  $u(t, x)$  can be expressed as the expectation of a star-product functional, which is nothing but a probabilistic solution constructed by making use of the below-mentioned branching particle systems and branching models. Let

$$M_{\star}^{\langle u_0, f \rangle}(\omega) = \prod_{\star} \star_{[x_m]} \Xi_{m_2, m_3}^{m_1} [u_0, f](\omega), \quad (5)$$

be a probabilistic representation in terms of tree-based star-product functional with weight  $(u_0, f)$ . For the details of the definition, see the succeeding sections. On the other hand,  $M_{\star}^{\langle U, F \rangle}(\omega)$  denotes the corresponding  $\star$ -product functional with weight  $(U, F)$ . In fact, as to be seen in what follows, in a similar manner as the case of a star-product functional we can construct a  $(U, F)$ -weighted tree-based  $\star$ -product functional  $M_{\star}^{\langle U, F \rangle}(\omega)$ , which is indexed by the nodes  $(x_m)$  of a binary tree. Here we suppose that  $U = U(x)$  (resp.  $F = F(t, x)$ ) is a non-negative measurable function on  $D_0$  (resp.  $\mathbb{R}_+ \times D_0$ ) respectively, and also that  $F(\cdot, x) \in L^1(\mathbb{R}_+)$  for each  $x$ . Indeed, in construction of the  $\star$ -product functional, the product in question is taken as ordinary multiplication  $*$  instead of the star-product  $\star$  in the definition of star-product functional.

**Theorem 1.** *Suppose that  $|u_0(x)| \leq U(x)$  for  $\forall x$  and  $|\tilde{f}(t, x)| \leq F(t, x)$  for  $\forall t, x$ , and also that for some  $T > 0$  ( $T \gg 1$  sufficiently large),*

$$E_{T,x}[M_{\star}^{\langle U, F \rangle}] < \infty, \quad \text{a.e.} - x \quad (6)$$

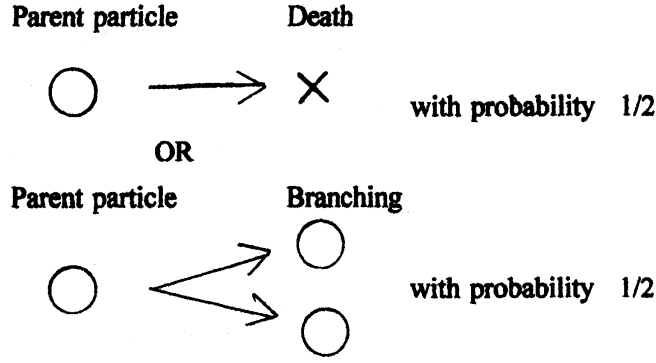
*Then there exists a  $(u_0, f)$ -weighted tree-based star  $\star$ -product functional  $M_{\star}^{\langle u_0, f \rangle}(\omega)$ , indexed by a set of node labels accordingly to the tree structure which a binary critical branching process  $Z^{K_x}(t)$  determines. Furthermore, the function*

$$u(t, x) = E_{t,x}[M_{\star}^{\langle u_0, f \rangle}] \quad (7)$$

*gives a unique solution to the integral equation (1). Here  $E_{t,x}$  denotes the expectation with respect to a probability measure  $P_{t,x}$  as the time-reversed law of  $Z^{K_x}(t)$ .*

## §3. Branching model

In this section we consider a continuous time binary critical branching process  $Z^{K_x}(t)$  on  $D_0$ , whose branching rate is given by a parameter  $\lambda|x|^2$ , whose branching mechanism is binary with equi-probability, and whose descendant branching particle behavior (or distribution) is determined by the kernel  $K_x$ . Next, taking



⊠ 1: Binary Branching

notice of the tree structure which the process  $Z^{K_x}(t)$  determines, we denote the space of marked trees

$$\omega = (t, (t_m), (x_m), (\eta_m), m \in \mathcal{V}) \quad (8)$$

by  $\Omega$ . Furthermore, we write the time-reversed law of  $Z^{K_x}(t)$  being a probability measure on  $\Omega$  as  $P_{t,x} \in \mathcal{P}(\Omega)$ . Here  $t$  denotes the birth time of common ancestor, and the particle  $x_m$  dies when  $\eta_m = 0$ , while it generates two descendants  $x_{m1}, x_{m2}$  when  $\eta_m = 1$ . On the other hand,

$$\mathcal{V} = \bigcup_{\ell \geq 0} \{1, 2\}^\ell$$

is a set of all labels, namely, finite sequences of symbols with length  $\ell$ , which describe the whole tree structure given. For  $\omega \in \Omega$  we denote by  $\mathcal{N}(\omega)$  the totality of nodes being branching points of tree, and let  $N_+(\omega)$  be the set of all nodes  $m$  being a member of  $\mathcal{V} \setminus \mathcal{N}(\omega)$ , whose direct predecessor lies in  $\mathcal{N}(\omega)$  and which satisfies the condition  $t_m(\omega) > 0$ , and let  $N_-(\omega)$  be the same set as described above, but satisfying  $t_m(\omega) \leq 0$ . Finally we put

$$N(\omega) = N_+(\omega) \cup N_-(\omega). \quad (9)$$

#### §4. Star-product functional and \*-product functional

In what follows we shall introduce a tree-based star-product functional in order to construct a probabilistic solution to the class of integral equations (1). First of all, we denote by the symbol  $\text{Proj}^z(\cdot)$  a projection of the objective element

onto its orthogonal part of the  $z$  component in  $\mathbb{C}^3$ , and we define a  $\star$ -product of  $\beta, \gamma$  for  $z \in D_0$  as

$$\beta \star_{[z]} \gamma = -i(\beta \cdot e_z) \text{Proj}^z(\gamma). \quad (10)$$

We shall define  $\Theta^m(\omega)$  for each  $\omega \in \Omega$  realized as follows. When  $m \in N_+(\omega)$ , then  $\Theta^m(\omega) = \tilde{f}(t_m(\omega), x_m(\omega))$ , while  $\Theta^m(\omega) = u_0(x_m(\omega))$  if  $m \in N_-(\omega)$ . Then we define

$$\Xi_{m_2.m_3}^{m_1}(\omega) \equiv \Xi_{m_2.m_3}^{m_1}[u_0, f](\omega) := \Theta^{m_2}(\omega) \star_{[x_{m_1}]} \Theta^{m_3}(\omega), \quad (11)$$

where as for the product order in the star-product  $\star$ , when we write  $m \prec m'$  lexicographically with respect to the natural order  $\prec$ , the term  $\Theta^m$  labelled by  $m$  necessarily occupies the left-hand side and the other  $\Theta^{m'}$  labelled by  $m'$  occupies the right-hand side by all means. And besides, as abuse of notation we write

$$\Xi_{m,\emptyset}^\emptyset(\omega) \equiv \Xi_{m,\emptyset}^\emptyset[u_0, f](\omega) := \Theta^m(\omega), \quad (12)$$

especially when  $m \in \mathcal{V}$  is a label of single terminal point in the restricted tree structure in question.

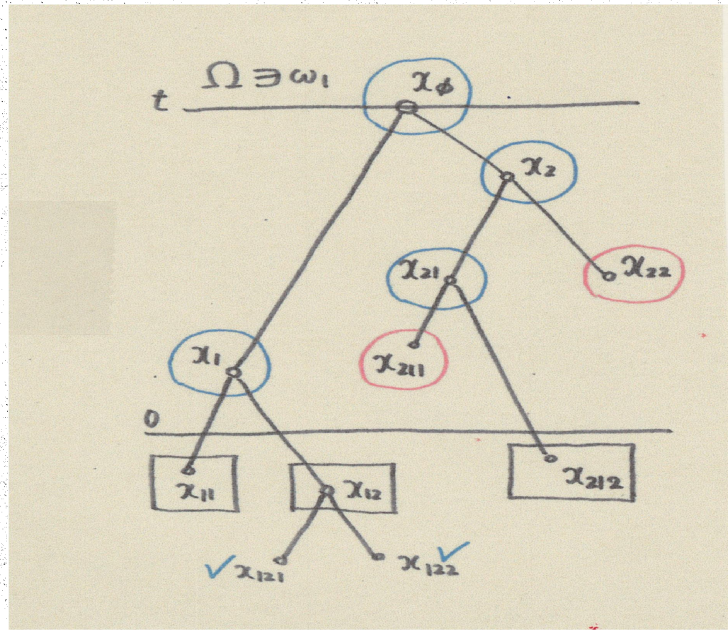


图 2: Example: A realized Tree

Under these circumstances, we consider a random quantity which obtained by executing the star-product  $\star$  inductively at each node in  $\mathcal{N}(\omega)$ , and we call it a tree-based  $\star$ -product functional, and we express it symbolically as

$$M_{\star}^{(u_0, f)}(\omega) = \prod \star_{[x_{\tilde{m}}]} \Xi_{m_2.m_3}^{m_1}[u_0, f](\omega), \quad (13)$$

where  $m_1 \in \mathcal{N}(\omega)$  and  $m_2, m_3 \in N(\omega)$ , and by the symbol  $\prod \star$  (as a product relative to the star-product) we mean that the star-products  $\star$ 's should be successively executed in a lexicographical manner with respect to  $x_{\tilde{m}}$  such that  $\tilde{m} \in \mathcal{N}(\omega) \cap \{|\tilde{m}| = \ell - 1\}$  when  $|m_1| = \ell$ .

**Example 1.** Now let us suppose that a tree structure  $\omega_1 (\in \Omega)$  has been realized here (see Figure 2).

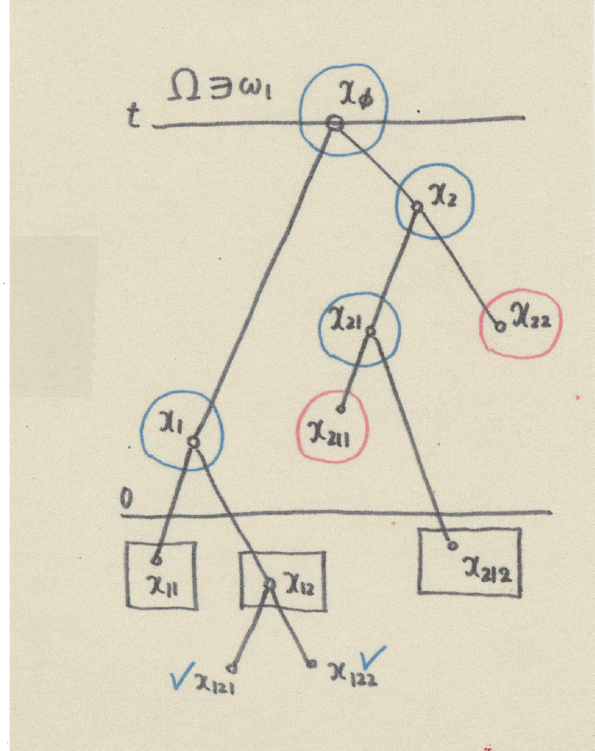


Fig 3: Classification of Nodes

Next we shall classify those nodes in the realized tree  $\omega_1$ . As a matter of fact, as to those two particles located at  $x_{11}$  and  $x_{12}$  with nodes of the level  $|m| = \ell = 2$  accompanied by the pivoting node  $x_1$ , we can construct

$$\Xi_{11,12}^1(\omega_1) = \Theta^{11}(\omega_1) \star_{[x_1]} \Theta^{12}(\omega_1)$$

by a star-product  $u_0(x_{11}(\omega_1)) \star_{[x_1]} u_0(x_{12}(\omega_1))$  in accordance with the rule, because both  $m_1 = 11$  and  $m_2 = 12$  lie in  $N_-(\omega)$ . As to the node  $x_{21}$ , how to construct  $\Xi(\omega_1)$  is the almost same thing as described above. In fact, it goes similarly because  $x_{211}$  lies in  $N_+(\omega_1)$  and  $x_{212}$  lies in  $N_-(\omega_1)$ . According to the rule, it follows that

$$\Theta^{211}(\omega_1) = \tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1)) \quad \text{and} \quad \Theta^{212}(\omega_1) = u_0(x_{212}(\omega_1)),$$

hence  $\Xi_{211,212}^{21}(\omega_1)$  is given by  $\tilde{f}(t_{211}(\omega_1), x_{211}(\omega_1)) \star_{[x_{21}]} u_0(x_{212}(\omega_1))$ , see Figure 3. Consequently, we obtain finally

$$M_{\star}^{(u_0, f)}(\omega_1) = (u_0(x_{11}) \star_{[x_1]} u_0(x_{12})) \star_{[x_\phi]} \left\{ \left( \tilde{f}(t_{211}, x_{211}) \star_{[x_{21}]} u_0(x_{212}) \right) \star_{[x_2]} \tilde{f}(t_{22}, x_{22}) \right\}. \quad (14)$$

□

### §5. Outline of proof: tree-based star-product functional as a solution

In this section we are first going to construct a  $(U, F)$ -weighted tree-based  $\star$ -product functional  $M_{\star}^{(U, F)}(\omega)$ , which is indexed by the nodes  $(x_m)$  of a binary tree. Here recall that  $U = U(x)$  (resp.  $F = F(t, x)$ ) is a non-negative measurable function on  $D_0$  (resp.  $\mathbb{R}_+ \times D_0$ ) respectively, and also that  $F(\cdot, x) \in L^1(\mathbb{R}_+)$  for each  $x$ . Moreover, in construction of the functional, the product is taken as ordinary multiplication  $*$  instead of the star-product  $\star$ .

In what follows we shall give an outline of the proof of Theorem 1. We need the following technical lemma, which plays an essential role in the proof.

**Lemma 2.** *For  $0 \leq t \leq T$  and  $x \in D_0$ , the function  $V(t, x) = E_{t,x}[M_{\star}^{(U, F)}(\omega)]$  satisfies*

$$e^{\lambda t |x|^2} V(t, x) = U(x) + \int_0^t ds \frac{\lambda |x|^2}{2} e^{\lambda s |x|^2} \left\{ F(s, x) + \int V(s, y) V(s, z) K_x(dy, dz) \right\}. \quad (15)$$

As a matter of fact, the mapping :  $[0, T] \ni t \mapsto e^{\lambda |x|^2 t} V(t, x) \in \bar{\mathbb{R}}_+$  is non-decreasing, so that, it proves to be that

$$E_{t,x}[M_{\star}^{(U, F)}(\omega)] < \infty \quad (16)$$

holds for  $\forall t \in [0, T]$  and  $x \in E_c$ , where  $E_c$  is a measurable set on which the validity of  $E_{t,x}[M_{\star}^{(U, F)}] < \infty$  may be kept. Another important aspect for the proof consists in establishment of the following  $M_{\star}$ -control inequality. That is to say, we have

$$|M_{\star}^{(u_0, f)}(\omega)| \leq |M_{\star}^{(U, F)}(\omega)| \quad (17)$$

because of the validity of a simple inequality

$$|w \star_{[x]} v| \leq |w| \cdot |v| \quad \text{for } w, v \in \mathbb{C}^3 \quad \text{and } x \in D_0.$$

On the other hand, it is derived that the space of solutions to (1) is formed by the condition

$$\int_0^T ds \int |u(s, y)| \cdot |u(s, z)| K_x(dy, dz) < \infty \quad \text{for } x \in E_c.$$

A similar discussion as above leads to

$$u(t, x) = E_{t,x}[M_{\star}^{(u_0, f)}(\omega)] = e^{-\lambda t|x|^2} u_0(x) + \int_0^t ds \lambda |x|^2 e^{-\lambda(t-s)|x|^2} \times \\ \times \frac{1}{2} \left\{ \tilde{f}(s, x) + \iint E_{s,x_1}[M_{\star}] \star_{[x]} E_{s,x_2}[M_{\star}] K_x(dx_1, dx_2) \right\}. \quad (18)$$

Finally we can deduce that  $u(t, x) = E_{t,x}[M_{\star}^{(u_0, f)}(\omega)]$  satisfies the integral equation (1), and this  $u(t, x)$  is a solution lying in the space  $\mathcal{D}$ . Actually,  $\mathcal{D}$  is a space of all functions  $\varphi : \mathbb{R}_+ \times D_0 \rightarrow \mathbb{C}^3$ , being continuous in  $t$  and measurable such that

$$\int_0^\infty ds \int |p(s, x, y; \varphi)| K_x(dy, dz) < \infty, \quad \text{a.e. } -x.$$

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